Some remarks on Marczewski-measurable sets and functions

Marcin Kysiak

Institute of Mathematics, Warsaw University

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Marczewski measurable sets and functions

Definition

- a set $X \subseteq \mathbb{R}$ is Marczewski measurable ($X \in (s)$ for short) if $\forall P \in \mathbf{Perf}\ \exists Q \subseteq P\ \ Q \in \mathbf{Perf} \wedge (Q \subseteq X \vee Q \cap X = \emptyset)$,
- a set $X \subseteq \mathbb{R}$ is Marczewski null ($X \in (s_0)$ for short) if $\forall P \in \mathbf{Perf}\ \exists Q \subseteq P\ \ Q \in \mathbf{Perf} \wedge Q \cap X = \emptyset$,
- a function $f : \mathbb{R} \to \mathbb{R}$ is Marczewski measurable, if it is measurable with respect to the σ-field (s).

Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is Marczewski measurable if, and only if, for every $P \in \mathbf{Perf}\ \exists Q \subseteq P\ \ Q \in \mathbf{Perf} \wedge f \restriction Q$ is continuous.

Indicatrices

Definition

Let $f : X \to Y$ be a function. The indicatrix $s(f) : Y \to \mathbf{Card}$ of the function f is defined by

$$s(f)(y) = |f^{-1}[\{y\}]|.$$

We say that $f, g : [0,1] \to [0,1]$ are equivalent ($f \sim g$, for short), if $s(f) = s(g)$.

Remark

The functions $f, g : [0,1] \to [0,1]$ are equivalent if, and only if, there exists a bijection $\varphi : [0,1] \to [0,1]$ such that $f \circ \varphi = g$.

General question

Let $\mathcal{F} \subseteq [0,1]^{[0,1]}$ be a class functions. Can we characterize functions equivalent to a member of $\mathcal{F}$?

In other words, can we characterize indicatrices of members of $\mathcal{F}$?

Some answers

Theorem (Morayne–Ryll-Nardzewski)

A function $f : [0,1] \to [0,1]$ *is equivalent to a Lebesgue measurable one (and equivalently, to a Baire-measurable one) if, and only if,* $s(f) > 0$ *on a perfect set or there exists* $y \in [0,1]$ *such that* $s(f)(y) = \mathfrak{c}$.

Moreover:

- Komisarski, Michalewski and Milewski characterized indicatrices of Borel functions (under Projective Determinacy),
- Kwiatkowska characterized indicatrices of continuous functions (in ZFC).

What about Marczewski-measurable functions?

Theorem

A function $f : [0,1] \to [0,1]$ *is equivalent to a Marczewski-measurable one if, and only if,* $s(f) > 0$ *on a perfect set or there exists* $y \in [0,1]$ *such that* $s(f)(y) = \mathfrak{c}$.

Corollary

Every Marczewski measurable function is equivalent to a Lebesgue measurable one (and to a Baire measurable one) and vice versa.

More details

Definition

A σ-algebra $\mathcal{A}$ has the weak Continuous Restriction Property if for every $\mathcal{A}$-measurable function $f : [0,1] \to [0,1]$ there exists a perfect set $P \subseteq [0,1]$ such that $f \restriction P$ is continuous.

Lemma

If a σ-algebra $\mathcal{A}$ has the weak Continuous Restriction Property then for each $\mathcal{A}$-measurable function $f : [0,1] \to [0,1]$ either $s(f) > 0$ on a perfect set or $s(f)$ takes value $\mathfrak{c}$.

Yet more details

Lemma

Assume that a σ-algebra $\mathcal{A}$ contains all Borel sets and that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Then

- *if a function $f : [0,1] \to [0,1]$ is constant on a set of cardinality $\mathfrak{c}$ then it is equivalent to an $\mathcal{A}$-measurable function,*
- *if a function $f : [0,1] \to [0,1]$ contains a perfect set in its range then it is equivalent to an $\mathcal{A}$-measurable function.*

Generality

Remark

The theorem is also true for other algebras, e.g. for algebras associated with "tree forcing notions".

Reference

- Marcin Kysiak, *Some remarks on indicatrices of measurable functions*, Bull. Polish Acad. Sci. Math. **53** (2005), 281-284.
- http://www.mimuw.edu.pl/∼mkysiak/